



TITLE:

# GO-spaces and orderability of compactifications (Research of Set-Theoretic and Geometric Topology and Their Applications)

AUTHOR(S):

Tanaka, Yoshio

---

CITATION:

Tanaka, Yoshio. GO-spaces and orderability of compactifications (Research of Set-Theoretic and Geometric Topology and Their Applications). 数理解析研究所講究録 2001, 1188: 127-137

ISSUE DATE:

2001-02

URL:

<http://hdl.handle.net/2433/64696>

RIGHT:

## GO-spaces and orderability of compactifications

東京学芸大学 田中祥雄 (Yoshio Tanaka)

*In this paper, we give some characterizations for certain compactifications of GO-spaces to be orderable by means of "cuts" in GO-spaces.*

Let  $(X, \leq)$  be a linearly ordered set. Then, a *linearly ordered topological space* (abbreviated LOTS) is a triple  $(X, \tau(\leq), \leq)$ , where  $\tau(\leq)$  is the usual *order topology* (i.e., open-interval topology) by the order  $\leq$ . Also,  $(X, \tau, \leq)$ ,  $\tau$  is a topology on  $X$ , is a *generalized ordered space* (abbreviated GO-space) if (i)  $\lambda(\leq) \subset \tau$ ; and (ii) every point of  $X$  has a local  $\tau$ -base consisting of (possibly degenerate) intervals of  $X$ . For a space  $(X, \tau)$ , there exists a linear order  $\leq$  of  $X$  such that  $(X, \tau, \leq)$  is a GO-space iff it is a (closed) subspace of a LOTS; see [L].

A space  $X$  is *orderable* (resp. *suborderable*) if  $X$  is homeomorphic to a LOTS (resp. GO-space) [N]. Thus, a space  $X$  is orderable iff the topology of  $X$  coincides with the order topology by some linear order of  $X$  [VRS].

For a space  $X$ , a compactification  $c(X)$  of  $X$  is a compact space such that  $X$  is homeomorphic to a dense subset of  $c(X)$ . We call a compactification  $c(X)$  of  $X$  *orderable* if the topology of  $c(X)$  is the order topology by some order of  $c(X)$ .

If no confusion, for a GO-space (or LOTS)  $(X, \tau, \leq)$ , we shall omit " $\tau$ " or " $\leq$ ". Also, we shall sometimes use "LOTS" (resp. "GO-spaces") instead of "orderable spaces" (resp. "suborderable spaces").

Let  $(X, \tau, \leq)$  be a GO-space. Let us consider a space  $Y$  containing a subspace  $(X, \tau)$  such that the order  $\leq$  on  $X$  can be so extended to some linear order  $\preceq$  on  $Y$  as to yield the given topology of  $Y$  as the order topology  $\tau(\preceq)$  by  $\preceq$ . Then, we say that  $Y$  is a *linearly ordered extension* of  $X$ . Also, let us call a compactification  $c(X)$  of  $X$  a *linearly ordered compactification* of  $X$  if  $c(X)$  is a linearly ordered extension of  $X$ . (When a GO-space  $(X, \tau, \leq)$  is homeomorphic to a dense subspace  $D$  of  $c(X)$  under a map  $f$ , we shall consider a GO-space  $(D, f(\tau), \leq_f)$  instead of  $(X, \tau, \leq)$ , here  $f(\tau) = \{f(G) : G \in \tau\}$ , and  $d <_f d'$  if  $x < x'$  for  $d = f(x)$ ,  $d' = f(x')$ ).

Let  $(X, \tau, \leq)$  be a GO-space, and  $\lambda = \tau(\leq)$  be the order topology on  $X$  defined by  $\leq$ . Let  $R = \{x \in X : [x, +\infty) \in \tau - \lambda\}$ , and  $L = \{x \in X :$

$(-\infty, x] \in \tau - \lambda\}$ , and  $Z$  be the set of all integers.

Define subsets  $X^*$  and  $\widetilde{X}$  of  $X \times Z$  as follows. Let  $X^*$ ;  $\widetilde{X}$  be a LOTS having the order topology by the lexicographic order on  $X^*$ ;  $\widetilde{X}$  respectively.

$$\begin{aligned} X^* &= (X \times \{0\}) \cup \{\langle x, n \rangle : x \in R, n < 0\} \cup \{\langle x, m \rangle : x \in L, m > 0\}. \\ \widetilde{X} &= (X \times \{0\}) \cup (R \times \{-1\}) \cup (L \times \{1\}). \end{aligned}$$

If  $X$  is not a LOTS, then  $\widetilde{X}$  is not a subspace of  $X^*$ , and  $X^*$  is not a linearly ordered extension of  $\widetilde{X}$  under the natural correspondence. For  $X^*$ ;  $\widetilde{X}$ , see [L] (or [N]); [MK] respectively.

*Remark 1.* (1) For a GO-space  $X$ ,  $X^*$  (resp.  $\widetilde{X}$ ) is a minimal (in the sense of inclusion) linearly ordered extension of  $X$  containing  $X$  as a closed (resp. dense) subset [L] (resp. [MK]).

(2) For the Sorgenfrey line  $S$ ,  $S$  is separable, and perfect (i.e., every closed subset is a  $G_\delta$ -set), and so is  $\widetilde{S}$ . But,  $S^*$  is neither separable nor perfect ([L]).

We note that there exists a perfect, GO-space  $X$ , but  $X$  has no perfect, linearly ordered extensions containing  $X$  as a closed or dense subset ([MK]).

(3) For a GO-space  $X$ , if  $X$  is metrizable; first countable; locally compact; paracompact, then so is  $X^*$  respectively ([L]). But,  $\widetilde{X}$  need not be metrizable even if  $X$  is a discrete, GO-space ([MK]).

Let  $(X, \leq)$  be a linearly ordered set. A pair  $(A|B)$  of subsets of  $X$  is called a *cut* of  $X$ , if  $X = A \cup B$ ,  $A \neq \emptyset$ ,  $B \neq \emptyset$ , and if  $x \in A$  and  $y \in B$ , then  $x < y$ .

For every cut  $(A|B)$  of  $X$ , exactly one of the following four cases arises. A cut  $(A|B)$  is a *jump* if it satisfies (C1), and a *gap* if it satisfies (C4); see ([E]). Note that, for cuts  $(A|B)$  and  $(C|D)$  of  $X$ ,  $A \subset C$  or  $C \subset A$ .

- (C1) There exist *Max*  $A$  and *min*  $B$ .
- (C2) There exists *Max*  $A$ , but no *min*  $B$ .
- (C3) There exists *min*  $B$ , but no *Max*  $A$ .
- (C4) There exists neither *Max*  $A$  nor *min*  $B$ .

Let  $(X, \leq)$  be a GO-space. A cut  $(A|B)$  of  $X$  is called a *pseudo-gap* if  $A$  and  $B$  are disjoint *open sets* satisfying (C2) or (C3); see [N]. We note that a GO-space  $(X, \leq)$  is a LOTS iff  $(X, \leq)$  has no pseudo-gaps.

Let  $(X, \leq)$  be a GO-space. Define a subset  $X^\sim$  of  $X \times \{0, \pm 1\}$  by

$$X^\sim = (X \times \{0\}) \cup \{\langle \text{Max} A, 1 \rangle : (A|B) \text{ is a pseudo-gap of } X \text{ having } \text{Max} A\} \cup \{\langle \text{min} B, -1 \rangle : (A|B) \text{ is a pseudo-gap of } X \text{ having } \text{min} B\}.$$

Let  $X^\sim$  be a LOTS having the order topology defined by the lexicographic order on  $X^\sim$ . Then  $\widetilde{X} = X^\sim$ .

For a LOTS  $(X, \leq)$ , define

$$X^+ = X \cup \{c = (A|B) : c \text{ is a gap of } X\} \cup \{\pm\infty\}.$$

Let  $X^+$  be a LOTS having the order topology by a linear order  $\preceq$  on  $X^+$  as follows: (i) For a gap  $c = (A|B)$ ,  $a \prec c$  for all  $a \in A$ , and  $c \prec b$  for all  $b \in B$ ; and (ii) For gaps  $c = (A|B)$  and  $c' = (A'|B')$ ,  $c \prec c'$  if  $A \subset A'$  and  $A \neq A'$ . Also, let  $-\infty \prec x$  and  $x \prec +\infty$  for all  $x \in X$ , but put  $-\infty = \min X$  if  $\min X$  exists, and put  $+\infty = \max X$  if  $\max X$  exists.

For a GO-space  $(X, \leq)$ ,  $X^+$  is defined by the closure of  $X$  in  $(X^*)^+$ . Then,  $X^+ = (X^+, \tau(\preceq), \preceq)$  is a linearly ordered compactification of  $X$ . See [N; Example VIII.3].  $X^+$  is called *Dedekind compactification* of  $X$ .

Let  $(X^\sim)^+ = X^\sim \cup \{\langle \alpha, 0 \rangle : \alpha = (A|B) \text{ is a gap of } X\} \cup \{\langle \pm\infty, 0 \rangle\}$  be a subset of  $X^+ \times \{0, \pm 1\}$ . Let  $(X^\sim)^+$  be a LOTS having the order topology by the lexicographic order on  $(X^\sim)^+$ . Then,  $X^+ = (X^\sim)^+ = (\widetilde{X})^+$ , so  $X^+$  is a linearly ordered compactification of  $\widetilde{X}$ .

*Remark 2.* For a GO-space  $(X, \leq)$ , a compact LOTS  $\ell X$  was defined in [K1] as the *minimal* linearly ordered compactification of  $X$  in the following sense: For each linearly ordered compactification  $L$  of  $X$ , there exists a continuous map  $f : L \rightarrow \ell X$  such that  $f|X$  is the identity map on  $X$ . (In [K1],  $\ell X$  is used in the study on normality of products of GO-spaces and cardinals). We can assume that  $X^+ = \ell X$  (in view of [K1]).

For a space  $X$ , let us consider the following compactifications of  $X$ .

$\alpha(X)$ : Alexandroff's one-point compactification.

$\beta(X)$ : Stone-Čech compactification.

$X^+$ : Dedekind compactification, but  $X$  is a GO-space.

The following facts are well-known. See [E] or [N], for example.

**Fundamental Facts:** (1) Every GO-space is hereditarily (collectionwise) normal, and hereditarily countably paracompact.

(2) For a LOTS  $(X, \leq)$ ,  $X$  is compact  $\Leftrightarrow X$  has no gaps, and there exist  $\min X$  and  $\max X \Leftrightarrow$  For every  $A \subset X$ , there exists  $\sup A$ , here  $\sup \emptyset = \min X$ , and  $\sup X = \max X$ .

(3) For a LOTS  $(X, \leq)$ ,  $X$  is connected  $\Leftrightarrow X$  has no jumps and no gaps.

(4) For a GO-space  $(X, \tau, \leq)$ ,  $\tau = \tau(\leq)$  if  $X$  is compact or connected.

*Example 1.* (1) (i) Let  $X = (0, 1) \cup \{2\}$  be a space with the usual topology. Then,  $X$  is a GO-space which is the topological sum of LOTS  $(0, 1)$  and  $\{2\}$ .

But,  $X$  is not orderable.

(ii) None of the following subspaces of the Euclidean plane is suborderable: The circle  $S^1$ ; The square  $[0, 1] \times [0, 1]$ ; The space obtained from the topological sum of  $n$  ( $\geq 3$ ) many intervals  $[0, 1]$  by identifying all zero-points.

(2) The Sorgenfrey line and the Michael line are GO-spaces, but none of them is orderable (in view of [L]).

(3) (i) Let  $X = \{0\} \cup (1, 2]$  be a space with the usual topology. Hence  $X$  is a GO-space, but not a LOTS by the usual order. While,  $X$  is orderable by the usual order  $\leq$ , but let  $x < 0$  for all  $x \in (1, 2]$ .

(ii) Let  $Y = ([0, \omega_1], \leq)$ , where  $\leq$  is the usual order. Let  $\tau$  be the topology on  $Y$  obtained from the order topology by isolating every countable limit ordinal. Then,  $(Y, \tau, \leq)$  is a GO-space, but not a LOTS. While,  $(Y, \tau)$  is orderable by the lexicographic order on  $([0, \omega_1) \times \mathbb{Z}) \cup \{(\omega_1, 0)\}$  ([L]).

(4) (i) Let  $X$  be the unit square  $[0, 1] \times [0, 1]$ , and define the order topology on  $X$  by the the lexicographic order. Then, as is well-known,  $X$  is a first countable, compact, connected LOTS, but  $X$  is not separable, hence not metrizable.

(ii) Let  $Y$  be  $[0, 1] \times \{0, 1\}$ , and define the order topology on  $Y$  by the lexicographic order. Then, as is well-known,  $Y$  is a first countable, compact, separable LOTS, but  $Y$  is not metrizable.

*Remark 3.* (1) Related to (1) of Example 1, the following modifications hold: (i) Let  $Y$  be a topological sum of a connected LOTS  $(X, \leq)$  and a point  $p$ . Then  $Y$  is suborderable, and  $Y$  is orderable iff  $\max X$  or  $\min X$  exists. (ii) Any connected space  $X$  with  $|X| \geq 2$  is not orderable if  $X - \{p\}$  is connected for any point  $p \in X$ , or  $X - \{q\}$  has at least three components for some point  $q \in X$ .

(2) Let  $X$  be suborderable. Then  $X$  is orderable if  $X$  is a topological group ([LiSaT]), or  $X$  is a metrizable space which is totally disconnected (i.e., any connected subset of  $X$  is a singleton).

(3) ([VRS]) If  $X \times Y$  is suborderable, then  $X$  is totally disconnected, or  $Y$  is discrete. Conversely, for any orderable (resp. suborderable) space  $X$ ,  $X \times Y$  is so respectively if  $Y$  is discrete. While, even if  $X \times Y$  is orderable with  $Y$  discrete,  $X$  need not be orderable. (In fact, let  $X$  be the space  $(0, 1) \cup \{2\}$  in Example 1(1), and let  $Y$  be a countably infinite discrete space).

**Proposition 1.** Let  $X$  be a GO-space. If  $X$  is separable metrizable, then  $X^*$ , and  $X^+$  are separable metrizable, hence so is  $\bar{X}$ .

**Corollary 2.** Let  $(X, \leq)$  be a GO-space. If  $X$  is separable metrizable, then  $X$  has at most countably many jumps and pseudo-gaps.

*Remark 4.* (1) Let  $X$  be a separable metrizable space. Then, as is well-known,  $\alpha(X)$  is metrizable if  $X$  is locally compact, but,  $\beta(X)$  is not even first countable if  $X$  is not compact.

(2) For a compactification  $Y$  of a space  $X$ , if  $Y$  is first countable, then  $|Y| \leq c = 2^\omega$  (thus,  $|X| \leq c$ ).

**Proposition 3.** For a LOTS  $(X, \leq)$ , the following are equivalent.

- (a)  $\alpha(X)$  is a linearly ordered compactification of  $(X, \leq)$ .
- (b) One of the following (i), (ii), and (iii) holds.
  - (i)  $X$  has no gaps, and there exists  $\min X$ , but no  $\max X$ .
  - (ii)  $X$  has no gaps, and there exists  $\max X$ , but no  $\min X$ .
  - (iii)  $X$  has only one gap, and there exist  $\min X$  and  $\max X$ .
- (c)  $\alpha(X) = X^+$ .

*Remark 5.* The linearly ordered extension for  $\alpha(X)$  in Proposition 3 is essential (by Example 2 below).

*Example 2.* Let  $N = \{1, 2, \dots\}$ . Let  $N$  be a LOTS  $(N, \leq)$  with the usual order  $\leq$ . Let  $X = (N, \preceq)$  be a LOTS, but the order  $\preceq$  is defined as follows:  $\dots \prec 4 \prec 2 \prec 1 \prec 3 \prec 5 \prec \dots$ . Then,  $\alpha(N) = N^+$ , but a linearly ordered compactification  $\alpha(X)$  of  $X = (N, \preceq)$  doesn't exist (by Proposition 3). While,  $N \cong X$ , so  $\alpha(N) \cong \alpha(X)$ , but  $N^+ \not\cong X^+$ . Hence,  $\alpha(X)$  is orderable, but  $\alpha(X) \not\cong X^+$ .

**Proposition 4.** ([VRS]) Let  $Y$  be a space having a dense subset  $X$ . If  $Y$  is suborderable, then the following hold.

- (1) If  $|X| \geq \omega$ , then the character  $\chi(Y) \leq |X|$ , and  $|Y| \leq 2^{|X|}$ .
- (2) If  $X$  is connected, then  $Y$  is connected and  $|Y - X| \leq 2$ .

The following lemma is shown by referring to [E; 6.3.2].

**Lemma 5.** (1) Let  $X$  be a separable connected, compact space. If  $X$  is orderable, then  $X$  is homeomorphic to the closed unit interval  $[a, b]$  in the Euclidean line  $\mathbf{R}$ .

(2) Let  $X$  be a separable connected space. If  $X$  is orderable, then  $X$  is homeomorphic to an interval of  $\mathbf{R}$ .

(3) Let  $X$  be a separable metrizable space. If  $X$  is suborderable, then  $X$  is homeomorphic to a subspace of  $\mathbf{R}$ .

*Remark 6.* (1) Not every separable compact LOTS is metrizable, also, not every compact connected LOTS is metrizable (by Example 1(4)).

(2) As is well-known, every separable suborderable space  $X$  is first countable, hereditarily separable, hereditarily Lindelöf, and  $|X| \leq 2^\omega$ .

*Remark 7.* (1) For a separable connected LOTS  $(X, \leq)$ ,  $X \cong \mathbf{R} \Leftrightarrow X$  has no Maximal point and no minimal point  $\Leftrightarrow X$  is a topological group.

(2) Let  $(K, +, \times)$  be a *field*, here  $(K, +)$  is an additive Abelian group, and  $(K, \times)$  is a multiplicative Abelian group with respect to  $K - \{0\}$ . Then,  $K$  with a linearly order  $\leq$  on  $K$  is called an *ordered field* if it is a LOTS  $(K, \tau(\leq), \leq)$  satisfying: For any  $a, b, c \in K$ ,  $a < b \Rightarrow a + c < b + c$ ; and  $a < b$  and  $c > 0 \Rightarrow a \times c < b \times c$ . An order field  $(K, \leq)$  is *Archimedean* if, for each  $a, b (> 0) \in K$ , there exists  $n \in \mathbf{N}$  with  $a < n \times b$ . Every Archimedean order field is a separable metrizable LOTS, thus it is homeomorphic to a subspace of  $\mathbf{R}$  (by Lemma 5(3)).

Let  $(K, \tau(\leq), \leq)$  be an ordered field. For  $x \in K$ , define the *absolute value*  $|x|$  by  $|x| = x$  if  $x \geq 0$ , and  $|x| = -x$  if  $x < 0$ . Then,  $\{V_\varepsilon(a) : a, \varepsilon \in K \text{ with } \varepsilon > 0\}$  is a base for the order topology  $\tau(\leq)$ , here  $V_\varepsilon(a) = \{x \in K : |x - a| < \varepsilon\}$ . For a function  $f : K$  (or  $[a, b] \subset K$ )  $\rightarrow K$ , using absolute values, the following can be defined by the same way as in  $\mathbf{R}$ :  $f$  is *bounded*, *continuous*, *differentiable*, or *integrable*.

Let  $K = (K, \tau(\leq), \leq)$  be an ordered field. Let us say that  $K$  is a *real number field* if it has no gaps (i.e.,  $K$  is connected). As is well-known, every real number field is isomorphic, hence, homeomorphic to  $\mathbf{R}$  (by (1)). We know many equivalent conditions for  $K$  to be  $\mathbf{R}$  (for example, every upper bounded subset  $A$  of  $K$  has  $\sup A$ ). Besides, we have the following equivalences by means of cuts of  $K$ . Here, a *map* means a *continuous* function defined on a closed interval  $[a, b]$  in  $K$ .

(Theorem): ([T2]) For an ordered field  $K$ ,  $K$  is  $\mathbf{R} \Leftrightarrow$  Any *map* to  $K$  is bounded and  $K$  is Archimedean  $\Leftrightarrow$  Any *map* to  $\mathbf{R}$  is bounded  $\Leftrightarrow$  For any *map*  $f$  to  $K$  (or  $\mathbf{R}$ ),  $f([a, b])$  has the Maximal (minimal) value  $\Leftrightarrow$  For any *map*  $f$  to  $K$  (or  $\mathbf{R}$ ),  $f([a, b]) = [f(a), f(b)]$  if  $f(a) \leq f(b) \Leftrightarrow$  Any differentiable *map* to  $K$  satisfies the Rolle's theorem  $\Leftrightarrow$  Any bounded *map* to  $K$  is integrable.

**Proposition 6.** For a space  $X$ , the following are equivalent.

- (a)  $X$  is a locally separable, metrizable, suborderable space.
- (b)  $X$  is the topological sum of subspaces of  $\mathbf{R}$ .

**Proposition 7.** Let  $X$  be a separable connected space, and let  $c(X)$  be a compactification of  $X$ . Then, (a)  $\Leftrightarrow$  (b), and (b)  $\Rightarrow$  (c) hold.

- (a)  $c(X)$  is orderable.
- (b)  $c(X) \cong [0, 1]$ .
- (c)  $X$  is homeomorphic to an interval of  $\mathbf{R}$ , and  $|c(X) - X| \leq 2$

*Remark 8.* The implication (c)  $\Rightarrow$  (a) (or (b)) in Proposition 7 doesn't hold. (In fact, put  $c(\mathbf{R}) = \alpha(\mathbf{R})$ , then  $|c(\mathbf{R}) - \mathbf{R}| = 1$ , but  $c(\mathbf{R}) \cong S^1$  is not orderable (by Example 1(1))).

**Lemma 8.** ([Sh]) Let  $(X, \leq)$  and  $(Y, \preceq)$  be connected LOTS. For a homeomorphism  $f : X \cong Y$ , (a) or (b) below holds.

- (a) For all  $x, y \in X$ ,  $x < y$  iff  $f(x) \prec f(y)$ .
- (b) For all  $x, y \in X$ ,  $x < y$  iff  $f(y) \prec f(x)$ .

**Theorem 9.** ([Sh]) Let  $X$  be a connected LOTS, and let  $c(X)$  be a compactification of  $X$ . Then the following are equivalent.

- (a)  $c(X)$  is orderable.
- (b)  $c(X) \cong X^+ (= X \cup \{\pm\infty\})$

*Remark 9.* (1) The connectedness of  $X$  in Theorem 9 is essential. (In fact, for a case  $c(X) = \alpha(X)$  (resp.  $c(X) = \beta(X)$ ), see Example 2 (resp. Example 3(2))).

(2) Let  $c(X)$  be a linearly ordered compactification of a connected LOTS  $X$  such that  $|c(X) - X| \leq 2$ . If  $c(X) = \beta(X)$ , then  $c(X)$  is orderable (by Corollary 19), however, if  $c(X) = \alpha(X)$ ,  $c(X)$  need not be orderable (by Remark 8).

For the following lemma, refer to [GJ], [E], or [T1]. Recall that a space  $X$  has countable tightness (abbreviated  $t(X) \leq \omega$ ) if, whenever  $x \in clA$ , there exists a countable subset  $C$  of  $A$  with  $x \in clC$ .

**Lemma 10.** (1) Let  $X$  be a normal space. If  $X$  is not countably compact, then  $\beta(X) - X$  contains a copy of  $\beta(\mathbb{N})$  as well as  $\beta(\mathbb{N}) - \mathbb{N}$ .

(2)  $\beta(\mathbb{N})$  is neither hereditarily normal nor hereditarily countably paracompact, in particular,  $\beta(\mathbb{N})$  is not orderable. Also,  $|\beta(\mathbb{N})| = 2^c$  ( $c = 2^\omega$ ), and  $t(\beta(\mathbb{N})) > \omega$ .

**Lemma 11.** For a suborderable space  $X$ , as is known, the following hold.

- (1) If  $X$  is countably compact, then  $X$  is sequentially compact.
- (2) If  $t(X) \leq \omega$ , then  $X$  is first countable, thus, every countably compact subset is closed.

**Proposition 12.** ([VRS]) Let  $\beta(X)$  be orderable. Then  $X$  is countably compact, hence sequentially compact.

**Corollary 13.** Let  $\beta(X)$  be orderable. Then  $X$  is compact if (a) or (b) below holds. (For  $F$ -spaces and  $P$ -spaces, see [GJ]).

- (a)  $\beta(X)$  has countable tightness.
- (b)  $X$  satisfies one of the following properties: Paracompact space; Realcompact space; Separable space;  $F$ -space;  $P$ -space.

For a GO-space  $(X, \leq)$ , define a subset  $X^\sharp$  of  $X^+ \times \{0, \pm 1\}$  by the following. Let  $X^\sharp$  be a LOTS having the order topology defined by the lexicographic



order on  $X^\sharp$ .

$X^\sharp = (X \times \{0\}) \cup \{\langle \text{Max}A, 1 \rangle : (A|B) \text{ is a pseudo-gap of } X \text{ having } \text{Max}A\} \cup \{\langle \text{min}B, -1 \rangle : (A|B) \text{ is a pseudo-gap of } X \text{ having } \text{min}B\} \cup \{\langle c, 1 \rangle, \langle c, -1 \rangle : c = (A|B) \text{ is a gap of } X\} \cup \{\langle \pm\infty, 0 \rangle\}$ .

Namely,  $X^\sharp = \widetilde{X} \cup \{\langle c, 1 \rangle, \langle c, -1 \rangle : c = (A|B) \text{ is a gap of } X\} \cup \{\langle \pm\infty, 0 \rangle\}$ . Obviously, if  $X$  has no gaps,  $X^\sharp = X^+$ . If  $X$  has a gap, then  $X^\sharp$  is not *minimal* (in the sense of Remark 2).

**Proposition 14.** Let  $(X, \leq)$  be a GO-space. Then the following hold.

- (1)  $X^\sharp$  and  $X^+$  are linearly ordered compactifications of  $X$ , as well as  $\widetilde{X}$ .
- (2)  $X^\sharp$  is connected  $\Leftrightarrow \beta(X)$  is connected  $\Leftrightarrow X$  is connected. While,  $X^+$  is connected  $\Leftrightarrow X$  has no jumps and no pseudo-gaps.
- (3)  $X^\sharp$  is metrizable  $\Leftrightarrow X$  is a separable metrizable space having at most countably many gaps  $\Leftrightarrow X$  is a separable metrizable space with  $|X^\sharp - X| \leq \omega$ .

**Lemma 15.** For a countably compact GO-space  $(X, \leq)$ , the following (1) and (2) hold.

- (1) For every continuous real-valued function  $f$  on  $X$ , there exist  $a, b \in X$  with  $a \leq b$  such that  $f$  is constant on  $R_b = \{x \in X : x \geq b\}$ , and on  $L_a = \{x \in X : x \leq a\}$ .
- (2) Every continuous real-valued function  $f$  on  $X$  can be continuously extendable over  $X^\sharp$  (hence,  $\beta(X) \cong X^\sharp$ ).

(In fact, for (1), assuming  $X$  has no Maximal point, we show that each real valued function  $f$  on  $X$  is constant on some  $R_b$  as in the poof of the Vickery's result on the ordinal space  $[0, \omega_1]$  (see [D; p.81], etc.). For (2), note that for a cut  $c = (A|B)$  of  $X$ ,  $A$  and  $B$  are clopen in  $X$  (so, they are countably compact GO-spaces) if  $c$  is a gap, a pseudo-gap, or a jump. Then, using (1), we can define a continuous extension  $F$  of  $f$  over  $X^\sharp$  naturally).

**Theorem 16<sup>1</sup>.** Let  $(X, \leq)$  be a GO-space. Then following are equivalent.

- (a)  $\beta(X)$  is orderable.
- (b)  $X$  is countably compact (equivalently, sequentially compact).
- (c)  $\beta(X) \cong X^\sharp$ .
- (d)  $\beta(X)$  is a linear ordered compactification of  $X$ .
- (e)  $\beta(\widetilde{X})$  is orderable with  $\beta(X) \cong \beta(\widetilde{X})$ .
- (f)  $\beta(X) \cong \beta(\widetilde{X}) \cong X^\sharp$ .

<sup>1</sup>S. Purisch [P] (resp. R. Kaufman [Ka]) has already proved that the equivalence (a)  $\Leftrightarrow$  (b) for a GO-space (resp. LOTS) holds by a different proof.

**Corollary 17.** For a GO-space  $X$ , let  $R(X) = \beta(X) - X$  be the remainder of  $\beta(X)$ . Then the following are equivalent.

- (a)  $\beta(X)$  is orderable.
- (b)  $R(X)$  is suborderable.
- (c)  $R(X)$  is hereditarily normal.
- (d)  $R(X)$  is hereditarily countably paracompact.
- (e)  $R(X)$  contains no copy of  $\beta(\mathbb{N})$ .

**Corollary 18.** For a GO-space  $X$ ,  $\beta(X)$  is orderable if  $R(X)$  satisfies one of the following properties:  $|R(X)| < 2^c$ ;  $t(R(X)) \leq \omega$ ; Each point of  $R(X)$  is a  $G_\delta$ -set in  $R(X)$ .

**Corollary 19.** For a connected LOTS  $X$ , the following are equivalent.

- (a)  $\beta(X)$  is orderable.
- (b)  $|R(X)| \leq 2$ .
- (c)  $\beta(X) \cong X^+ (= X \cup \{\pm\infty\})$ .

**Corollary 20.** For a GO-space  $X$ ,  $\beta(X) \cong X^+ \Leftrightarrow X$  is a countably compact space with  $X^\# \cong X^+$ .

**Corollary 21.** For GO-spaces  $(X, \leq)$  and  $(Y, \preceq)$  with  $X \cong Y$ , if  $X$  is countably compact, then  $X^\# \cong Y^\#$ .

*Remark 10.* (1) In Theorem 16, even if  $\beta(\widetilde{X})$  is orderable with  $\beta(\widetilde{X}) \cong X^\#$ ,  $\beta(X)$  need not be orderable (by Example 3(1)). Note that, for a GO-space  $Y$ , if  $\beta(Y)$  is orderable, then so is  $\beta(\widetilde{Y})$ , but the converse doesn't hold.

(2) In Corollary 17, we can replace " $R(X)$ " by " $\beta(X)$ ". We can't omit "hereditarily" in (c) and (d). (In fact, let  $X$  be a GO-space which is locally compact, in particular, connected, but  $X$  is not countably compact).

(3) (i) Related to Corollary 18, as a special case, the following holds<sup>2</sup>.

For  $|R(X)| = 1$ ,  $X$  is *orderable* iff  $\beta(X)$  is orderable. But, for  $|R(X)| = 2$ , the "if" part need not hold (by Example 3(2)).

(ii) For Corollary 18, even if  $|R(X)| = 2^c$ ,  $\beta(X)$  need not be orderable. (In fact, let  $X$  be a GO-space which is separable, but not compact).

The author has the following question<sup>3</sup>: Is there a GO-space  $X$  such that  $|R(X)| = 2^c$ , but  $\beta(X)$  is orderable (equivalently,  $X$  is countably compact)?

(4) In Corollary 19, for the implications (a)  $\Rightarrow$  (b) or (c), and (b)  $\Rightarrow$  (c), the connectedness of  $X$  is essential (by Example 3(2)). Also, in Corollary 21, the countable compactness of  $X$  is essential (by Example 2).

<sup>2</sup>K. Miyazaki announced this fact (with a different proof).

<sup>3</sup>N. Kemoto gave an affirmative answer to this question (in general, for any cardinal  $\kappa$  with  $cf\kappa \geq \omega_2$ , there exists a countably compact GO-space  $X$  with  $|R(X)| = \kappa$ ) in [K2].

*Example 3.* (1) Let  $X = [0, 1] \cup (2, 3]$ . Then  $X$  is a GO-space by the usual topology (also,  $X$  is orderable). Then,  $\widetilde{X} = X^\# = X^+ = X \cup \{\langle 1, 1 \rangle\}$  is a compact LOTS. Thus,  $\beta(\widetilde{X}) = X^\#$  is orderable. But,  $\beta(X)$  is not orderable (by Proposition 11).

(2) Let  $\Omega$  be the Long line; that is,  $\Omega$  is a space  $(\Omega, \tau(\leq), \leq)$  obtained by replacing all jumps in the ordinal space  $[0, \omega_1)$  by the closed intervals, where  $\leq$  is the obvious order. Then  $\Omega$  is a connected and countably compact LOTS, but  $\Omega$  is neither separable nor compact. Also,  $\alpha(\Omega) \cong \beta(\Omega) \cong \Omega^+ = \Omega \cup \{+\infty\}$  (by means of Lemma 15).

(i) Define  $(-\Omega)$  by a LOTS  $(\Omega, \tau(\leq'), \leq')$ , but  $\leq'$  is defined as follows:  $x' <' x$  if  $x < x'$ . Let  $\Sigma = (-\Omega) \cup \Omega$  be a LOTS defined by an order  $\preceq$ :  $x \prec x'$  if  $x <' x'$  in  $(-\Omega)$ ,  $y \prec y'$  if  $y < y'$  in  $\Omega$ , and  $x \prec y$  if  $x \in (-\Omega)$  and  $y \in \Omega$ . Then,  $\Sigma$  is a countably compact, connected space having no Maximal point and no minimal point. Let  $T$  be the topological sum of  $(\Sigma, \preceq)$  and a point  $+\infty$ . Let  $\tau$  be the topology of the space  $T$ , and define the obvious order  $\preceq'$  of  $T$  with the Maximal point  $+\infty$ . Then  $(T, \tau, \preceq')$  is a countably compact GO-space which is not orderable, and  $\beta(T) \cong T^+ = T \cup \{\pm\infty\} \cup \{\langle +\infty, -1 \rangle\}$  (hence,  $|R(T)| = 2$ ).

(ii) For  $n \in N$  ( $n \neq 1$ ), let  $X$  be the topological sum of  $n$  many LOTS  $(\Sigma, \preceq)$ . Then  $X$  is a countably compact disconnected LOTS having gaps but no jumps. Then,  $X^+$  is a connected space with  $|X^+ - X| = n + 1$ . While,  $\beta(X)$  is a disconnected space with  $|\beta(X) - X| = 2n$ . Thus,  $\beta(X)$  is orderable such that  $|\beta(X) - X| = 2n$  ( $|X^+ - X| = n + 1$ ), but  $\beta(X) \not\cong X^+$ .

(iii) Let  $\Gamma = \Omega \cup (-\Omega)$  be a LOTS defined by a similar way as  $\Sigma$ . Then  $\Gamma$  is a countably compact space having only one gap  $\omega_1 = (\Omega | (-\Omega))$  and no jumps. Then,  $\Gamma^+ = \Gamma \cup \{\omega_1\}$  is connected. While,  $\beta(\Gamma) \cong \Gamma \cup \{\langle \Omega, \pm 1 \rangle\}$  is disconnected. Hence,  $\beta(\Gamma)$  is orderable, but  $\beta(\Gamma) \not\cong \Gamma^+$ .

*Acknowledgement.* The author would like to thank Professors N. Kemoto and T. Miwa for their valuable suggestions.

## REFERENCES

- [D] J. Dugundji, Topology, Allyn and Bacon, Inc., Boston, 1967.
- [E] R. Engelking, General Topology, PWN-Polish Scientific Pub. Warszawa, 1977.
- [GJ] L. Gillman and M. Jerison, Rings of continuous functions, van Nostrand, Princeton, 1960.
- [Ka] R. Kaufman, Ordered sets and compact spaces, Colloquim Math., 17(1967), 35-39.

- [K1] N. Kemoto, Normality of products of GO-spaces and cardinals, *Topology Proc.*, 18(1993), 133-142.
- [K2] N. Kemoto, personal communication.
- [LiSaT], C. Liu, M. Sakai, and Y. Tanaka, Orderability of topological groups and biradial spaces, to appear in *Questions and Answers in General Topology*, 19(2001).
- [L] D. J. Lutzer, On generalized ordered spaces, *Dissertationes Math.*, Warszawa, 1971, 1-36.
- [MK] T. Miwa and N. Kemoto, Linearly ordered extensions of GO-spaces, *Topology and Appl.*, 54(1993), 133-140.
- [N] J. Nagata, *Moderan General Topology*, North-Holland, Amsterdam, Newyork, Oxford, 1983.
- [P] S. Purisch, On the orderability of Stone-Čech compactifications, *Proc. Amer. Math. Soc.*, 1973, 55-56,
- [Sh] T. Shinoda, Linearly ordered topological spaces and their generalization, Master-thesis (Tokyo Gakugei univ.) 2000, 1-55.
- [T1] Y. Tanaka, On closedness of  $C$ - and  $C^*$ -embeddings, *Pacific J. Math.*, 68(1977), 283-292.
- [T2] Y. Tanaka, Ordered fields and the axiom of continuity, *Bull. Tokyo Gakugei Univ.*, Sect. 4, 46(1994), 1-6.
- [VRS] M. Venkataraman, M. Rajagopalan and T. Soundararajan, Orderable spaces, *Gen. Topology and Appl.*, 2(1972), 1-10.

Department of Mathematics, Tokyo Gakugei University, Koganei, Tokyo, 184-8501, JAPAN

*e-mail address:* ytanaka@u-gakugei.ac.jp